

STABILITY OF HAMILTON'S CIGAR SOLITON

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ABSTRACT. In this paper, we shall use the Kähler geometry formulation to study the global behavior of the Ricci flow on R^2 . The geometric feature of our Ricci flow is that it has finite width. Our aim is to determine the limiting metric (which corresponds an eternal Ricci flow) obtained by L.F.Wu. We can use the classification result of Daskalopoulos-Sesum to give a sufficient condition such that the limiting metric of L.F.Wu is the metric of cigar soliton.

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1. INTRODUCTION

In this paper, we study the stability of Hamilton's cigar soliton of the Ricci flow in R^2 :

$$(1) \quad \partial_t g(t) = -2Rc(g(t)),$$

with the initial metric $g(0)$ of bounded curvature. The local existence of the flow on a complete non-compact Riemannian manifold has been proved by W.Shi in [10] (and see [11] for the local derivative estimates via the maximum principle method). Ricci flow on R^2 can be considered as a degenerate diffusion [3]. In the works [9] and [12], there are interesting results about the global Ricci flow on R^2 . In particular, in [12], the global Ricci flow on R^2 is proved provided the initial complete metric has bounded curvature and bounded quantity $|d \log tr_{g_E} g(0)|_{g(0)}$, where g_E is the standard Euclidean metric on R^2 (and we shall always these two conditions are true). Furthermore, when the initial metric has positive curvature and finite circumference at infinity, then the flow converges to the cigar metric at the time infinity. Recall here that the *circumference at infinity* of the Ricci 2-manifold (R^2, g) , which is defined as

$$C_\infty(g) = \sup_K \inf_D \{L(\partial D); \text{ for any compact } K \text{ and open } D, K \subset D \subset R^2\}.$$

One may find more references about recent development of Ricci flow in R^2 in [4] [5] [7] [8]. The aim of this paper is to extend the above result of L.F. Wu without the positive curvature assumption.

We now define the width of the evolving metric $g = g(t)$. Let $F : R^2 \rightarrow [0, \infty)$ be a proper function F , such that $F^{-1}(a)$ is a compact subset of R^2

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for every $a \in [0, \infty)$. The width of F is defined to be the supremum of the lengths of the level curves of F , namely,

$$w(F) = \sup_c L(F = c).$$

The width $w(g)$ of the metric g is defined to be the infimum

$$w(g) = \inf_F w(F).$$

We now recall some well-known formulae. In dimension two, the Ricci curvature of the Riemannian manifold (M^2, g) is

$$\text{Ric}(g) = Kg,$$

where $K = K(g)$ is the Gauss curvature of the metric g . Let $R(g(t))$ be the scalar curvature of the metric $g(t)$. Then $R(g) = 2K(g)$ and the Ricci flow in dimension two is reduced to the conformal flow

$$\partial_t g(t) = -R(g(t))g(t).$$

We are interested in the Ricci flow on $M^2 = R^2$. Given a metric g_0 on R^2 and let $g(t) = u(t)g_0$ with $u(t) > 0$ being a smooth function in R^2 . Then

$$R(g(t)) = u(t)^{-1}(-\Delta_{g_0} \log u(t) + R_0),$$

where Δ_{g_0} is the Laplacian operator of the metric g_0 with Analytist's sign and R_0 is the scalar curvature of g_0 . We simply write $u = u(t)$, $R = R(g(t))$, etc, when the notations cause no confusion. In this case, the Ricci flow can be written as

$$(2) \quad u_t = -Ru, \text{ in } R^2; \quad u(0) > 0,$$

which again can be written as

$$(3) \quad \partial_t u = \Delta_{g_0} \log u(t) - R_0, \text{ in } R^2, \quad u(0) > 0.$$

Taking the time derivative of (2) we find that

$$R_t = \Delta_g R + R^2.$$

Note also that the area element changes at the rate

$$\frac{d}{dt} dA(g(t)) = -R dA(g(t)).$$

Recall that the global existence of the Ricci flow under suitable geometric conditions has been studied by L.F.Wu [12]. To handle the convergence of the flow at time infinity, we need the following concept of convergence.

Definition 1. *The Ricci flow $g(t)$ is said to have modified subsequence convergence, if there exists a 1-parameter family of diffeomorphisms $\{\Phi(t_j)\}$ ($t_j \rightarrow \infty$) such that there exists a subsequence (denoted again by t_j) such that the sequence $\Phi(t_j)^*g(t_j)$ converges uniformly on every compact set as $t_j \rightarrow \infty$.*

We then state the following result of L.F. Wu [12] (see the main theorem in page 440 there).

Theorem 2. *Let $g(t) = e^{\tilde{u}(t)}g_E$ be a solution to (1) such that $g(0) = e^{\tilde{u}_0}g_E$ is a complete metric with bounded curvature and $|\nabla\tilde{u}_0|$ is uniformly bounded on R^2 . Then the Ricci flow has modified subsequence convergence as $t_j \rightarrow \infty$ with the limiting metric g_∞ being complete metric on R^2 . Furthermore, the limiting metric is the cigar soliton if $C_\infty(g(0)) < \infty$ and $g(0)$ has positive curvature.*

We point out that the diffeomorphisms $\Phi(t_j)$ used in Theorem 2 are of the special form

$$\Phi(t)(a, b) = (e^{\frac{-\tilde{u}(x_0, t)}{2}}a, e^{\frac{-\tilde{u}(x_0, t)}{2}}b) = (x_1, x_2) = x,$$

where $x_0 = (0, 0)$. The important fact for these diffeomorphisms is that

$$|\nabla_{g(t)}f(x, t)| = |\nabla_{\Phi(t)^*g(t)}f((a, b), t)|.$$

for any smooth function f and $x = \Phi(t)(a, b)$. Note that in the convergent part of Theorem 2, the limiting metric corresponds an eternal Ricci flow $g_\infty(t)$ on R^2 . In fact, it is proved in Theorem 2.4 in [12] that the curvature and $|\nabla\tilde{u}|^2$ of the global flow are uniformly bounded. According to the classification result [4] [5], the limiting flow is cigar soliton provided the width of g_∞ is finite, i.e., $w(g_\infty) < \infty$. Our aim is to show this is true in certain circumstances.

We shall assume that there is a potential function $f(0)$ for the initial metric $g(0)$ in the sense that $R(0) = \Delta_{g(0)}f(0)$ in R^2 . It is clear that the potential function plus any constant is still a potential function. For the cigar metric $g_c = \frac{dx^2+dy^2}{1+x^2+y^2}$, we have its potential function $f_0 = \log(1+x^2+y^2)$ in R^2 . the cigar metric has finite width, i.e., $w(g_c) < \infty$. We shall take g_0 the cigar metric in our new result.

We shall prove the following result.

Theorem 3. *Assume that the Riemannian manifold $(R^2, g(0))$ has bounded curvature and bounded quantity $|d \log \text{tr}_{g_E} g(0)|_{g(0)}$ in R^2 . Assume also that $g(0) = u(0)g_c$, where g_c is the metric of cigar soliton and $u(0)$ is positive function in R^2 such that $\log u(0)$ is also a bounded function (so $g(0)$ has finite width). Suppose that the potential function $f(0)$ of $g(0)$ satisfies that $f_0 - f(0) \in L^\infty(R^2, g(0))$. Then the Ricci flow with the initial metric $g(0)$ has global solution with its limiting metric at $t = \infty$ the cigar soliton metric.*

Our result is different from the result of Hsu [9], which is recoded in [7]. The result above can be considered as the stability property of cigar soliton. The global existence of the Ricci flow (and the limiting metric) follows from the work of L.Wu [12]. We just need to show that the width $w(g(t))$ is uniformly bounded. We shall use the Kähler-Ricci flow formulation [8] and the maximum principle trick [10] to prove theorem above. One of the important observation in our argument of the result above is that for the function $v = f(0) - f$, it satisfies that

$$(4) \quad (\partial_t - \Delta)v = -\Delta f(0) = -u^{-1}u(0)R(0), \text{ in } R^2 \times (0, \infty)$$

with the initial data $v(0) = 0$ in R^2 .

The structure of this paper is as follows. In the section 2, we recall Kähler geometry formulation of the 2-d ricci flow and prove some a priori estimates. In the section 3, we give the proof of theorem 3. Some well known facts about cigar metric is recalled in the last section.

2. KÄHLER-RICCI FLOW FORMULATION

We shall use the Kähler-ricci flow formulation to study the Ricci flow on R^2 . We shall consider the Ricci flow (2) as the Kähler-Ricci flow by setting

$$g_{i\bar{j}} = g_{0i\bar{j}} + \partial_i \partial_{\bar{j}} \phi,$$

where $\phi = \phi(t)$ is the Kähler potential of the metric $g(t)$ relative to the metric g_0 . Note that

$$g(0)_{i\bar{j}} = g_{0i\bar{j}} + \partial_i \partial_{\bar{j}} \phi_0,$$

In this situation, the Ricci flow can be written as

$$(5) \quad \partial_t \phi = 4 \log \frac{g_{01\bar{1}} + \phi_{1\bar{1}}}{g_{01\bar{1}}} - 4f_0, \quad \phi(0) = \phi_0,$$

where f_0 is the potential function of the metric g_0 in the sense that $R(g_0) = \Delta_{g_0} f_0$ in R^2 . Such a potential function has been introduced by R. Hamilton in [6]. We remark that the initial data for the evolution equation (5) is $\phi(0)$ which is non-trivial. Note also that we have used the factor 4 in right side of (5) which can allow us to use the usual Laplacian operator Riemannian geometry, for otherwise, we need to use the normalized Laplacian in Kähler geometry.

Let

$$f = -\partial_t \phi.$$

Then, taking the time derivative of (5), we have

$$(6) \quad \partial_t f = \Delta_g f, \quad f(0) = -\partial_t \phi(0).$$

By this we can easily get (4). The important fact for us is that

$$(7) \quad \Delta_g f = R.$$

To prove this, we recall some well-known results. Recall that any 2-d Riemannian manifold is a 1-d Kähler manifold. On a Kähler manifold of complex dimension n , the metric in coordinate expression is

$$g = g_{i\bar{j}} dz^i dz^{\bar{j}}$$

and the Ricci form is

$$\rho = \frac{\sqrt{-1}}{2} R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = -\sqrt{-1} \partial_i \partial_{\bar{j}} G dz^i \wedge dz^{\bar{j}}$$

with

$$G =: G(g) = \log \det(g_{i\bar{j}}).$$

Then the Ricci curvature of g is

$$R_{i\bar{j}} = -2\partial_i \partial_{\bar{j}} G$$

and its scalar curvature is

$$R(g) = 2g^{i\bar{j}}R_{i\bar{j}} = -\Delta_g G,$$

where $(g^{i\bar{j}})$ is the inverse of the matrix $(g_{i\bar{j}})$ and $\Delta_g = 4g^{i\bar{j}}\partial_i\partial_{\bar{j}}$ is the Laplacian operator of the metric g . In the complex dimension one case, we have

$$Kg_{1\bar{1}} = R_{1\bar{1}} = -2\partial_1\partial_{\bar{1}}G,$$

where K is the Gauss curvature of the metric g . Then the Ricci potential function is $f = -G$.

For the cigar metric

$$g_c = \frac{|dz|^2}{1 + |z|^2},$$

on R^2 , we have

$$g_{1\bar{1}} = \frac{1}{1 + |z|^2}, \quad g^{1\bar{1}} = 1 + |z|^2$$

and

$$G = -\log(1 + |z|^2).$$

Then $R = \frac{4}{1+|z|^2}$, $K = \frac{2}{1+|z|^2}$, and

$$Kg_{1\bar{1}} = \frac{2}{(1 + |z|^2)^2} = 2\partial_1\partial_{\bar{1}}\log(1 + |z|^2).$$

Hence

$$f(z) = -G = \log(1 + |z|^2)$$

is the Ricci potential of the metric g .

We now prove (7). Note that

$$Rg_{1\bar{1}} = 2R_{1\bar{1}} = -4\partial_1\partial_{\bar{1}}G.$$

Applying $\partial_1\partial_{\bar{1}}$ to (5) we obtain that

$$-4\partial_1\partial_{\bar{1}}f = 4\partial_1\partial_{\bar{1}}G - 4\partial_1\partial_{\bar{1}}G(g_0) - 4\partial_1\partial_{\bar{1}}f_0.$$

Taking the trace with respect to $g_{1\bar{1}}$ we have

$$-\Delta_g f = -R + \frac{1}{u}(R_0 - \Delta_{g_0}f_0) = -R,$$

which is (7). Here we have used that

$$(8) \quad \Delta_g := \Delta = u^{-1}\Delta_{g_0}$$

and

$$(9) \quad R_0 = \Delta_{g_0}f_0.$$

It is well-known [12] that R is uniformly bounded in any finite interval and $|f_t|$ and $|\nabla f|^2$ are bounded for each $t \geq 0$.

Using (8) and (9), we can write (3) as

$$(10) \quad (\log u - f_0)_t = \Delta(\log u - f_0), \quad \text{in } R^2.$$

We define w such that

$$u = e^{w-f+f_0} = e^{w+v+f_0+f(0)},$$

where

$$w = \log u - f_0 + f = \log u - v.$$

Then adding (6) and (10), we have

$$\partial_t w = \Delta_g w, \quad \text{in } R^2$$

with the initial data

$$w(0) = \log u(0) - f_0 + f(0), \quad \text{in } R^2,$$

which is a bounded function in R^2 by our assumption. By standard computations we have, in R^2 ,

$$(\partial_t - \Delta_g)w^2 = -2|\nabla w|^2$$

and

$$(\partial_t - \Delta_g)|\nabla w|^2 = -2|D^2 w|^2.$$

Then we have

$$(11) \quad (\partial_t - \Delta_g)(t|\nabla w|^2 + w^2) \leq 0.$$

To apply the maximum principle, we need to verify that $w(t)$ is bounded in the finite time interval $[0, T]$. First we note that

$$w_t = (\log u)_t + f_t = 2R, \quad \text{in } R^2,$$

which is uniformly bounded. Then $w(x, t) = w(x, 0) + 2 \int_0^t R$ is bounded in $R^2 \times [0, T]$ for $T > 0$ (and so about its derivatives). Hence we can use the maximum principle to conclude that

$$\sup_{R^2 \times [0, T]} |w| \leq \sup_{R^2} |w(x, 0)|.$$

With this understanding we use the Bernstein-Shi-Bando trick to (11) [11] to get the estimate that there is a uniform constant $C > 0$ such that

$$|\nabla w| \leq Ct^{-1}, \quad \text{in } R^2 \times (0, \infty).$$

This implies that w converges to a constant c .

We now can give an outline of the proof of Theorem 3: We shall show in next section that $f(0) - f = v$ is uniformly upper bounded. Once this is done, we then can conclude that

$$(12) \quad u(t) = e^{w+v-f(0)+f_0}$$

is uniformly upper bounded, which makes the metric $g(t)$ be uniformly upper bounded by a scale of the cigar soliton and the limiting metric g_∞ have finite width $w(g_\infty)$, which is the cigar soliton, by using the classification result of ancient solutions due to Daskalopoulos and Sesum [4] [5]. This will completes the proof of Theorem 3.

3. MAIN ESTIMATES AND THE PROOF OF THEOREM 3

In this section we shall give the uniform upper bound of v , which is enough for us to conclude the desired convergence in C_{loc}^2 . In fact the limiting metric g_∞ at time infinity can be obtained by the singularity analysis.

Recall now that $g_0 = g_c$ and $g(0) = u_0 g_c$, where g_c is the metric of the cigar soliton and $u_0 > 0$ is a bounded positive function in R^2 . For convenient of the readers, we recall some facts about the cigar soliton in R^2 in the appendix. Good references for this metric are [6] and [2]. The crucial fact for the cigar metric is that

$$C_\infty(g_c) < \infty.$$

Note that

$$R(0) = u_0^{-1}(-\Delta_{g_c} \log u_0 + R_c).$$

Then for $u = u(t)$,

$$R(0) = u_0^{-1}(-u \Delta_g \log u_0 + R_c).$$

Inserting this into (4) we get that

$$(\partial_t - \Delta)v = \Delta_g \log u_0 - u^{-1}u(0)R_c.$$

Let $h = v - \log u_0$. Then we have

$$(\partial_t - \Delta)h = -u^{-1}u(0)R_c < 0, \text{ in } R^2 \times (0, \infty), \quad h(0) = -\log u_0.$$

By assumption we have that $\log u_0$ is bounded. Then as above we can use the maximum principle to get the uniform upper bound for h . Then we know that $v = f(0) - f$ is uniformly upper bounded in R^2 , and hence $g(t)$ is uniformly controlled by the cigar metric g_c up to a factor. Therefore, the width

$$w(g(t)) < \infty$$

is uniformly bounded, and then

$$w(g_\infty) < \infty.$$

Then, the limiting metric is a cigar metric and we have completed the proof of Theorem 3.

4. APPENDIX: HAMILTON'S CIGAR SOLITON

Since the stability of cigar metric is our main topic here, we prefer to review a little more about it. Hamilton's cigar soliton is a special solution to the Ricci flow equation of metrics $g(t)$ in R^2 :

$$g_t = -Rg, \text{ in } R^2.$$

Namely, it is the one-parameter family of complete Riemannian metrics of the form

$$g_c(t) = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}$$

in R^2 . Define the diffeomorphism

$$\phi_t(a, b) = (e^{2t}a, e^{2t}b) = (x, y)$$

and

$$g_c(a, b) := \phi_t^* g_c(t) = \frac{da^2 + db^2}{1 + a^2 + b^2},$$

which is the so called the cigar metric.

We now re-write the cigar metric in polar coordinates (r, θ) with $r = \sqrt{a^2 + b^2}$ such that

$$g_c = \frac{dr^2 + r d\theta^2}{1 + r^2}.$$

One can compute that the scalar curvature of g_c is

$$R_c = \frac{4}{1 + r^2}$$

and its area element is

$$dv_c = \frac{r dr d\theta}{1 + r^2}.$$

Let $z = a + \sqrt{-1}b$. Then we have

$$g_c = \frac{|dz|^2}{1 + |z|^2},$$

which is the Kähler metric on $R^2 := \mathcal{C}$. Define the parameter s such that

$$ds = \frac{dr}{1 + r^2}.$$

Then we have

$$s = \log(r + \sqrt{1 + r^2}) = \operatorname{arcsinh} r$$

and

$$g_c = ds^2 + \tanh^2 s d\theta^2.$$

Then

$$R_c = 4(\cosh s)^{-2}.$$

We now try other method of finding the Ricci potential f_0 of the metric g_c , i.e.,

$$\frac{R_c}{2} g_c = \nabla^2 f_0.$$

We shall look for f being the radial function $f_0 = f_0(s)$. Define the local normal frame $e_1 = \frac{\partial}{\partial s}$ and

$$e_2 = \frac{1}{\tanh s} \frac{\partial}{\partial \theta}$$

. Then

$$\frac{R_c}{2} = \frac{R_c}{2} g_c(e_1, e_1) = \nabla^2 f(e_1, e_1) = f_0''(s)$$

and

$$\frac{R_c}{2} = \frac{R_c}{2} g_c(e_2, e_2) = -(\nabla_{e_2} e_2) f_0.$$

By these relations we find that

$$f_0 = 2 \log \cosh s.$$

Let $g_c = w_0(dx^2 + dy^2)$ in R^2 , where

$$w_0(x, y) = \frac{1}{1 + x^2 + y^2} = (\cosh s)^{-2}.$$

The key observation is that we have the relation

$$\log w_0 + f_0 = 0.$$

This fact plays a role in our formulation of Theorem 3.

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